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1986 J. Phys. A: Math. Gen. 19 L589

(http://iopscience.iop.org/0305-4470/19/10/006)

J. Phys. A: Math. Gen. 19 (1986) L589–L594. Printed in Great Britain

LETTER TO THE EDITOR

Statistical mechanical models for phase screens

Joseph F Benzoni†, Sarben Sarkar‡ and David Sherrington†§

† Department of Physics, Imperial College, London SW7 2BZ, UK
‡ Centre for Theoretical Studies, Royal Signals and Radar Establishment, Great Malvern, Worcs WR14 3PS, UK
§ Statistical Physics, Schlumberger-Doll Research, Ridgefield, CT 06877, USA

Received 22 January 1986

Abstract. On the basis of the strong link between limit theorems in probability theory and the mathematical formulation of critical phenomena, models for random phase screens are formulated in terms of Hamiltonian statistical mechanical systems. The probability distribution for the intensity of light scattered by the screen in the forward direction and far field is calculated in various cases. A criterion for the universal features contained in the intensity distribution is given.

The study of the scattering of waves from rough objects has been of growing interest recently. In two dimensions a canonical model of such scattering is given by a phase screen [1, 2]. This is a screen which instantaneously shifts the phase of a monochromatic wave impinging at $\boldsymbol{r}$ (a point on the screen) by $\phi(\boldsymbol{r})$. It is an idealisation of the consequence of passage through a transparent medium of refractive index different from that of the surroundings and of possibly variable thickness. We should note that this is to be contrasted with an amplitude screen, such as a diffraction grating, in which the amplitude is modulated but not the phase. The idealised phase screen leaves the amplitude unaffected.

It has been conjectured [3] that the form of the probability distribution of the intensity of the scattered wave falls into a limited set of classes. In separate work a fundamental connection [4, 5] has been established between generalised limit theorems in probability theory and scaling ideas of critical phenomena. It seems reasonable to believe that there might be a connection between these two types of universal behaviour. It is our purpose here to suggest such a connection.

In the usual formulations of phase screens [1, 2] the $\{\phi(\boldsymbol{r})\}$ are taken to be in a Gaussian distribution (but quenched for a particular screen). Since only phase differences are relevant a complete probabilistic specification of the screen can then be given by the correlation function $\langle\phi(\boldsymbol{r}+\boldsymbol{r}_0)\phi(\boldsymbol{r}_0)\rangle$ where $\langle\ \rangle$ refers to an average over the probability distribution. A power-law dependence of the correlation function on $|\boldsymbol{r}|$ denotes fractal behaviour.

With any probability distribution P it is possible to associate a 'Hamiltonian' H through

$$H=-\log P \tag{1}$$

on absorbing the 'temperature' into H. Statistical mechanics is the study of systems whose 'field' distribution is determined by equation (1) where H usually has some

0305-4470/86/100589+06\$02.50

microscopic physical origin. The study of such systems in the region of continuous phase transitions is a well developed branch of critical phenomena. Near such a transition the average correlations between fields at different spatial locations become long-ranged, whilst at the critical point the dependence on separation becomes power law. The analogy with the fractal phase screen is thus immediately suggested and it is natural therefore to enquire what statistical mechanics can tell one about phase screen expectations. Hence we will discuss systems characterised by Hamiltonians. We have freedom of choice of the functional relation between the fields (viz order parameters or 'spins') of the Hamiltonian and the phase shifts $\{\phi(\boldsymbol{r})\}$. We can also (in principle) consider a number of different forms for the interactions in H. We can thereby obtain a variety of models for phase screen distributions.

The usual phase screen model chooses for H a long-range interaction between ϕ at different points which is determined by the inverse correlation function. We shall choose models which in the Hamiltonian formulation have a simpler form but which have equal *a priori* validity. In the Hamiltonian models fractality is recovered only at critical points but, as is well known, critical exponents and appropriate distribution functions normally depend only on a limited number of features of the actual Hamiltonians, such as space and 'spin' dimensionalities. This behaviour is commonly known as universality. We shall see how this universality manifests itself in distributions of transmitted intensity in the phase screen problem, concentrating in the present letter on situations in which $\phi(\boldsymbol{r})$ is allowed only a finite number of discrete values. Away from the critical point the screen is smooth, non-fractal and not completely universal, but nevertheless useful results can be obtained from the analogy.

We shall find it convenient to discretise the space $\boldsymbol{r}$ for part of our specific analysis, but this is of no consequence in the regime of interest provided the 'lattice' spacing is taken to be small enough—it is well known to be irrelevant for critical behaviour where the characteristic correlation length is long and a similar situation applies to the screen (where one might envisage the lattice scale as related to the inner scale).

The first investigation [6] of phase screens which allowed for random discrete fluctuations in the phase was due to Jakeman and Hoenders (JH). There, a model of a one-dimensional phase screen was developed with

$$\phi(x)=\phi_0+\tilde{\phi}S(x) \tag{2}$$

where x is a position in the direction of variation across the screen, $S(x)$ is a quenched random variable taking values ± 1 and the crossings between $S(x)=\pm 1$ are Poisson distributed. Physically this corresponds to transmission through a thin transparent sheet with random rectangular corrugations

$$h(x)=h_0+hS(x) \tag{3}$$

$$\tilde{\phi}=k\Delta nh \tag{4}$$

where Δn is the refractive index difference and k is the wavevector. The discrete version of this has the screen consisting of elementary facets whose centre points are denoted by $\{x_i\}$. From a Hamiltonian point of view the Poisson distribution of JH is

$$H[\{S(x_i)\}]=\frac{W}{2\xi}-n_{\{S(x_i)\}}\log\frac{W}{2\xi}+\log\Gamma(n_{\{S(x_i)\}}+1) \tag{5}$$

where $n[\{S(x_i)\}]$ is the number of zero crossings in the configuration $\{S(x_i)\}$ with

$$n[\{-S(x_i)\}] = n[\{S(x_i)\}] \tag{6}$$

and W is the aperture of the screen.

From statistical mechanics we have learnt that critical behaviour is governed only by considerations such as the dimensionality of the system, the symmetry of H and whether the interaction is long- or short-ranged relative to some measure. The H in equation (5) has a symmetry group Z_2, i.e. it is invariant under the transformations

$$S(x_i) \rightarrow -S(x_i) \qquad \forall i. \tag{7}$$

In order to indicate the solvability possibilities offered by a Hamiltonian formulation and to illustrate the operation of universality we consider here other phase screen problems characterised by simple nearest-neighbour Hamiltonians, two with Z_2 symmetry, one with Z_3. These Hamiltonians are

$$H_1^{(2)} = -J \sum S_i S_j \tag{8}$$

$$H_2^{(2)} = -J \sum S_i' S_j' \tag{9}$$

$$H_3^{(3)} = -\sum S_i'' \mathbf{J}_{ij} S_j'' \tag{10}$$

where Σ denotes summation over nearest neighbours, the S_i take the values ± 1, the S_i' the values $\{0, \pm 1\}$, the S_i'' are 3-vectors $(1, 0, 0)$ and permutations, $\mathbf{J}_{ij}$ is given by

$$\mathbf{J}_{ij} = J \begin{pmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{pmatrix} \tag{11}$$

and the superscripts (n) indicate that the symmetry is Z_n. For $a = \frac{1}{2}$, $H_3^{(3)}$ is the three-state clock model. The Z_3 symmetry is immediately apparent for $H_3^{(3)}$ as written above but for application below it is convenient to re-express it in terms of S':

$$H_3^{(3)} = -J \sum [1 + \tfrac{1}{2}(1+a) S_i' S_j' + \tfrac{3}{2}(1+a) S_i'^2 S_j'^2 - (1+a)(S_i'^2 + S_j'^2)]. \tag{12}$$

We now consider the application to screens characterised by equation (2) with $S(x)$ now either two-valued, $S(x) = \pm 1$, or three-valued, $S(x) = 0, \pm 1$, and distributed according to the above Hamiltonians. Restricting discussion to transmission in the forward direction in the Fraunhofer (far-field) region, the standard Huygens–Fresnel integral representation yields for the normalised scattered wave ε

$$\varepsilon = W^{-1} \sum_j \exp[\mathrm{i}\tilde{\phi} S(x_j)] \tag{13}$$

where W is the number of discrete x points in the screen, $S(x)$ is S_i or S_i' as appropriate and the irrelevant overall phase shifts due to ϕ_0 and the unscattered phase shift have been ignored. The summation is over the width of the screen. Simplifications ensue from

$$\exp(\mathrm{i}\tilde{\phi} S_j) = \cos \tilde{\phi} + \mathrm{i} S_j \sin \tilde{\phi} \tag{14}$$

and

$$\exp(\mathrm{i}\tilde{\phi} S_j') = 1 + \mathrm{i} S_j' \sin \tilde{\phi} + (S_j')^2 (\cos \tilde{\phi} - 1) \tag{15}$$

leading naturally to formulation in terms of the block variables

$$S_{1,W} = W^{-1} \sum_j S_j \tag{16}$$

$$S'_{1,W} = W^{-1} \sum_j S'_j \tag{17}$$

$$S_{2,W} = W^{-1} \sum_j (S'_j)^2. \tag{18}$$

It is because of these simplifications that we discuss here phase shifts having a finite number of discrete values, rather than a continuum.

The probability distributions for the intensity I corresponding to (13) are

(a) for a system described by $H_1^{(2)}$, equation (8),

$$P_1^{(2)}(I) = \left.\frac{P(S)}{S \sin^2 \tilde{\phi}}\right|_{S=(I-\cos^2 \tilde{\phi})^{1/2}/\sin \tilde{\phi}} \tag{19}$$

where

$$P(S) = \langle \delta(S - S_{1,W}) \rangle \tag{20}$$

(b) for systems described by $H_2^{(2)}$, equation (9), and $H_3^{(3)}$, equation (12),

$$P_i^{(i)}(I) = \iint \mathrm{d}S_1\, \mathrm{d}S_2\, \delta\{I - S_1^2 \sin^2 \tilde{\phi} - [1 + S_2(\cos \tilde{\phi} - 1)]^2\} P^{(i)}(S_1, S_2) \tag{21}$$

where

$$P^{(i)}(S_1, S_2) = \langle \delta(S_1 - S'_{1,W}) \delta(S_2 - S_{2,W}) \rangle \qquad i = 2, 3. \tag{22}$$

The subscripts and superscripts correspond to those in equations (8), (9) and (12) while the angle brackets denote averaging with respect to the probability distribution e^{-H} where H is the corresponding Hamiltonian.

For the problems that we are considering it is the $P(S)$ and $P(S_1, S_2)$ which exhibit universality in an appropriate limit. Since the expressions for $P(I)$ depend on $\tilde{\phi}$ they are not themselves universal quantities, although they are made out of the universal quantities $P(S)$ and $P(S_1, S_2)$ in a direct way. It is easy to see that for a general system allowing for n discrete phase change possibilities joint probability distributions of block variables up to $(n-1)$th order need to be considered. Other finite symmetry groups can be treated similarly.

The analysis of the systems in equations (8)-(10) can proceed via the transfer matrix technique [7]. It is straightforward to show that in the thermodynamic limit, in which all other lengths become much greater than the discretisation length a, the distribution functions are characteristic functions of W/ξ where ξ is the correlation function given by the asymptotic relation for the average for an infinite system:

$$\langle \psi_i \psi_j \rangle = \langle (\psi_i)^2 \rangle \exp(-|x_i - x_j|/\xi a) \tag{23}$$

where ψ_i is a canonical spin. For W/ξ large there results the central limit Gaussian form, whereas for W/ξ small one obtains the following results.

(i) For the system characterised by $H_1^{(2)}$

$$P(S) = \tfrac{1}{2}\{(1 - W/2\xi)[\delta(1-S) + \delta(1+S)] + \tfrac{1}{2}(W/\xi)\theta(1-S^2)\} + \mathrm{O}[(W/\xi)^2] \tag{24}$$

where θ denotes the standard Heaviside step function. JH obtained an equivalent $P(S)$.

(ii) For $H_2^{(2)}$ and $H_3^{(3)}$ we find that

$$P^{(2)}(S_1, S_2) = \tfrac{1}{2}\delta(S_2 - 1)\{(1 - W/2\xi)[\delta(S_1+1) + \delta(S_1 - 1)] + \tfrac{1}{2}(W/\xi)\theta(1 - S_1^2)\} + \mathrm{O}[(W/\xi)^2]. \tag{25}$$

(In fact $P^{(2)}$ can be obtained exactly as in reference [6] but for comparison with $P^{(3)}$ below we prefer to exhibit only the perturbative form.)

Moreover

$$P^{(3)}(S_1, S_2)$$
$$\propto \tfrac{1}{3}[1-\tfrac{2}{3}(W/\xi)]\{\delta(S_1)\delta(S_2)+\delta(S_2-1)[\delta(S_1+1)+\delta(S_1-1)]\}$$
$$+\tfrac{1}{9}(W/\xi)\{\delta(S_2-1)\theta(1-S_1^2)-2[\delta(S_1+S_2)+\delta(S_1-S_2)]$$
$$\times[\theta(S_2-1)-\theta(S_2)]\}+\mathrm{O}[(W/\xi)^2]. \qquad (26)$$

For the Hamiltonians $H_1^{(2)}$, $H_2^{(2)}$ and $H_3^{(3)}$ we have

$$\xi=\tfrac{1}{2}\exp(2J) \qquad (27)$$

$$\xi=\tfrac{1}{4}\exp(2J) \qquad (28)$$

$$\xi=\tfrac{1}{3}\exp[J(1+a)] \qquad (29)$$

respectively. ξ large corresponds to the critical region, $J\to\infty$ (or $T\to 0$) in the present models. This is the regime of universality and consequently we would expect results identical to equations (24)-(26) for any short-range systems of the same symmetry. It is because both $H_1^{(2)}$ and $H_2^{(2)}$ have Z_2 symmetry that $P(S)$ is identical to $P^{(2)}(S)$ for the system described by $H_2^{(2)}$ where

$$P^{(i)}(S)=\int P^{(i)}(S_1, S_2)\,\mathrm{d}S_2 \qquad i=2,3. \qquad (30)$$

On the other hand, the Z_3 symmetry system, $H_3^{(3)}$, has $P^{(3)}(S)$ different from $P(S)$ and $P^{(3)}(S_1, S_2)\neq P^{(2)}(S_1, S_2)$. This leads to an intensity probability distribution which differs from that in the Z_2 symmetric theory. We thus see how the rigorous universal properties of distributions for the block spins imply a form of universality for the intensity distributions. The universality between systems can be further demonstrated by considering more general Hamiltonians such as the Blume-Emery-Griffiths Hamiltonian [8], which contains the models considered above as special cases.

In principle it is possible to extend the above ideas to space dimensions higher than one. The analysis is much harder and far from complete. It is, of course, possible to simulate the behaviour of the Hamiltonians in these dimensions on the computer and furthermore to take advantage of importance sampling techniques [9] such as that of Metropolis which have been developed for systems with few-body Hamiltonian interactions. In a future paper we will illustrate the applications of such a procedure to a two-dimensional telegraph screen. There are also possible approximate methods for analysis based on Wilson's approximate recursion relation [10] and the $P(S)$ extracted for a two-dimensional Ising system at criticality [11] is qualitatively in agreement with that obtained from Monte Carlo simulations [12]. The intensity distribution can then be inferred.

Finally, we comment further on why we have emphasised models in which the phase ϕ is allowed only a discrete number of values. This is for two main reasons. Firstly, for such situations $\exp(\mathrm{i}\phi_i)$ can be expressed as a finite polynomial in ϕ_i, requiring powers up to the $(n-1)$th for an n-valued system, thereby enabling the intensity distribution to be expressed in terms of a joint distribution of block 'spin' powers. Secondly, when reformulated in terms of continuous variables, such models yield effective non-Gaussian distributions with (in general) non-trivial critical behaviour; for example

$$\sum_{\{\sigma_i\}} \exp\left(\tfrac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j\right) \qquad \sigma_i = \pm 1$$

is equivalent to

$$\int \prod_i \frac{\mathrm{d}\psi_i}{\sqrt{2\pi}} \exp\left\{-\sum_i \frac{\psi_i^2}{2} + \sum_i \log\left[\cosh\left(\sum_j K_{ij}\psi_j\right)\right]\right\}$$

where

$$\sum_j K_{ij}K^T_{jk} = J^{-1}_{ik}$$

and ψ_i is an unrestricted continuous variable.

We are grateful to E Jakeman for discussions. One of us (JB) thanks RSRE, Malvern for the funding of his Research Assistantship at the Physics Department, Imperial College.

References

[1] Jakeman E 1983 *Opt. Acta* **30** 1207
[2] Berry M V 1979 *J. Phys. A: Math. Gen.* **12** 781
[3] Jakeman E and Pusey P 1978 *Phys. Rev. Lett.* **40** 546
[4] Cassandro M and Jona-Lasinio G 1978 *Adv. Phys.* **27** 913
[5] Kadanoff L P 1966 *Physics* **2** 263
[6] Jakeman E and Hoenders B J 1982 *Opt. Acta* **29** 1587
[7] Schultz T, Mattis D and Lieb E 1964 *Rev. Mod. Phys.* **36** 856
[8] Blume M, Emery V J and Griffiths R B 1971 *Phys. Rev.* A **4** 1071
[9] Binder K 1979 *Monte Carlo Methods in Statistical Physics* (Berlin: Springer)
[10] Wilson K G 1971 *Phys. Rev.* B **4** 3184
[11] Bruce A D 1981 *J. Phys. C: Solid State Phys.* **14** 3667
[12] Binder K 1981 *Z. Phys.* B **43** 119